SSU-HEP-10/7

HYPERFINE STRUCTURE OF THE GROUND STATE MUONIC ^{3}He ATOM

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On the basis of the perturbation theory in the fine structure constant α and the ratio of the electron to muon masses we calculate one-loop vacuum polarization and electron vertex corrections and the nuclear structure corrections to the hyperfine splitting of the ground state of muonic helium atom ($\mu\ e\ ^3_2He$). We obtain total result for the ground state hyperfine splitting $\Delta\nu^{hfs} = 4166.471$ MHz which improves the previous calculation of Lakdawala and Mohr due to the account of new corrections of orders α^5 and α^6. The remaining difference between our theoretical result and experimental value of the hyperfine splitting lies in the range of theoretical and experimental errors and requires the subsequent investigation of higher order corrections.

PACS numbers: 36.10.Gv, 12.20.Ds, 32.10.Fn

I. INTRODUCTION

Muonic helium atom ($\mu\, e\, ^3_2He$) represents the simple three-body atomic system consisting of one electron, negative charged muon and positive charged helion (^3_2He). Contrary to the muonic helium atom ($\mu\ e\ ^4_2He$) it has more complicated ground state hyperfine structure which appears due to the interaction of the magnetic moments of the electron, muon and helion [1–9]. The investigation of the energy spectrum of this three-particle bound state is important for the further check of quantum electrodynamics. Moreover, light muonic atoms (muonic hydrogen, muonic helium, ions of muonic helium etc.) represent a unique laboratory for precise determination of the nuclear properties such as the nuclear charge radius [10]. Hyperfine splitting (HFS) of the ground state of muonic helium atom ($\mu\ e\ ^3_2He$) was measured many years ago with sufficiently high accuracy [11]:

$$\Delta\nu^{hfs}_{exp} = 4166.3(2)\ MHz. \tag{1}$$

There are two approaches to the calculation of the energy spectrum of muonic helium atom ($\mu\ e\ ^3_2He$). First approach in [1, 2, 7] is based on the perturbation theory (PT) for the Schrödinger equation. In this case there is the analytical solution for the three particle bound state wave function in the initial approximation. Using it the analytical calculation of different corrections to HFS can be performed. Contrary to the energy levels of two-particle bound states which were accurately calculated in quantum electrodynamics [12–17], the hyperfine splitting of the ground state in muonic helium atom was calculated on the basis of the perturbation theory with essentially less accuracy. Another approach is built on the variatinal method [4–6, 8, 9, 18–20] which allows to increase the accuracy of the calculation. In the beginning, the accuracy of the HFS calculation was not sufficiently high because corrections of six order in α were estimated only approximately. A feature that

distinguishes light muonic atoms among the simplest atoms is that the structure of their energy levels strongly depends on the vacuum polarization, nuclear structure and recoil effects. Subsequently, the corrections of order α^2 to hyperfine splitting were studied in [5, 6] on the basis of variational and global-operator method. The theoretical value of HFS obtained in [5, 6] contains very small uncertainty and agrees with the experimental value (1).

In this work, which continues our investigation [21], we aim to refine the calculation of Lakdawala and Mohr [1] using their analytical approach to the description of the muonic helium atom. We investigate such contributions of the one-loop electron vacuum polarization of order $\alpha^5 M_e/M_\mu$ and the nuclear structure of orders α^5, α^6 which are significant for the improvement of the theoretical value of the hyperfine splitting obtained in [1] on the basis of perturbation theory. Another purpose of our study consists in the improved calculation of the electron one-loop vertex corrections to HFS of order α^5 using the analytical expressions of the Dirac and Pauli form factors of the electron.

The bound particles in muonic helium atom have different masses $m_e \ll m_\mu \ll m_\alpha$. As a result the muon and helion compose the pseudonucleus $(\mu\ {}^3_2He)^+$ and the muonic helium atom looks as a two-particle system in the first approximation. Three-particle bound system $(\mu\ e\ {}^3_2He)$ is described by the Hamiltonian [22, 23]:

$$H = H_0 + \Delta H + \Delta H_{rec}, \quad H_0 = -\frac{1}{2M_\mu}\nabla^2_\mu - \frac{1}{2M_e}\nabla^2_e - \frac{2\alpha}{x_\mu} - \frac{\alpha}{x_e}, \tag{2}$$

$$\Delta H = \frac{\alpha}{x_{\mu e}} - \frac{\alpha}{x_e}, \quad \Delta H_{rec} = -\frac{1}{m_h}\boldsymbol{\nabla}_\mu \cdot \boldsymbol{\nabla}_e, \tag{3}$$

where $\mathbf{x}_\mu$ and $\mathbf{x_e}$ are the coordinates of the muon and electron relative to the helium nucleus, $M_e = m_e m_h/(m_e + m_h)$, $M_\mu = m_\mu m_h/(m_\mu + m_h)$ are the reduced masses of subsystems $(e\ {}^3_2He)^+$ and $(\mu\ {}^3_2He)^+$. In the initial approximation the wave function of the ground state has the form [22–24]:

$$\Psi_0(\mathbf{x_e}, \mathbf{x}_\mu) = \psi_e(\mathbf{x_e})\psi_\mu(\mathbf{x}_\mu) = \frac{1}{\pi}(2\alpha^2 M_e M_\mu)^{3/2} e^{-2\alpha M_\mu x_\mu} e^{-\alpha M_e x_e}. \tag{4}$$

The hyperfine interaction in the ground state in $(\mu\ e\ {}^3_2He)$ is determined by the following Hamiltonian:

$$\delta H = -\frac{8\pi}{3}(\boldsymbol{\mu}_N \cdot \boldsymbol{\mu}_\mu)\delta(\mathbf{x}_\mu) - \frac{8\pi}{3}(\boldsymbol{\mu}_\mu \cdot \boldsymbol{\mu}_e)\delta(\mathbf{x}_\mu - \mathbf{x}_e) - \frac{8\pi}{3}(\boldsymbol{\mu}_e \cdot \boldsymbol{\mu}_N)\delta(\mathbf{x}_e), \tag{5}$$

where $\boldsymbol{\mu}_e = -g_e e/(2m_e)\mathbf{s}_e$, $\boldsymbol{\mu}_\mu = -g_\mu e/(2m_\mu)\mathbf{s}_\mu$, $\boldsymbol{\mu}_N = -g_N e/(2m_p)\mathbf{I}_N$ are magnetic moments of the electron, muon and helion. The total spin of the three spin-1/2 particles can be either 3/2 and 1/2. The matrix element of (5) leads to the shift of the energy levels which takes on form:

$$\delta E = < \delta H > = -a\ (\mathbf{I}_N \cdot \mathbf{s}_\mu) - b\ (\mathbf{s}_\mu \cdot \mathbf{s}_e) - c\ (\mathbf{s}_e \cdot \mathbf{I}_N), \tag{6}$$

where

$$a = \frac{2\pi\alpha}{3}\frac{g_N g_\mu}{m_p m_\mu} < \delta(\mathbf{x}_\mu) >,\ b = \frac{2\pi\alpha}{3}\frac{g_\mu g_e}{m_\mu m_e} < \delta(\mathbf{x}_\mu - \mathbf{x}_e) >,\ c = \frac{2\pi\alpha}{3}\frac{g_e g_N}{m_e m_p} < \delta(\mathbf{x}_e) > . \tag{7}$$

The diagonalization of the matrix element $< \delta H >$ gives three eigenvalues:

$$\nu_{1,2} = \frac{1}{4}(a+b+c) \pm \frac{1}{2}(a^2+b^2+c^2-ab-bc-ca)^{1/2},\ \nu_3 = -\frac{1}{4}(a+b+c). \qquad (8)$$

The values $\nu_{1,2}$ and ν_3 correspond to the total angular momentum $\frac{1}{2}$ and $\frac{3}{2}$. In the case of muonic helium ($\mu\ e\ ^3_2He$) we have the relations $a \gg b$ and $a \gg c$. So, the eigenvalues $\nu_{1,2}$ can be written with good accuracy in a more simple form:

$$\nu_1 = \frac{3}{4}a + \ldots, \quad \nu_2 = -\frac{1}{4}a + \frac{1}{2}(b+c) + \ldots. \qquad (9)$$

As a result the smaller hyperfine splitting interval related to the experiment (1) is given by

$$\Delta\nu^{hfs} = \nu_2 - \nu_3 = \frac{3}{4}(b+c) + O(\frac{b}{a}, \frac{c}{a}). \qquad (10)$$

Basic contributions to the coefficients a, b, c were calculated analytically in [1] from the contact interaction (5) in the first and second order PT. Taking into account numerical values of gyromagnetic factors $g_e = 2$ for the b coefficient, $g_e = 2(1+\kappa_e) = 2(1+1.1596521859\cdot 10^{-3})$ for the c coefficient, $g_\mu = 2(1+\kappa_\mu) = 2\cdot(1+1.16592069(60)\cdot 10^{-3})$, $g_N = 2\cdot 2.127497718(25)$, we obtain for them:

$$b_0 = \nu_F = \frac{8\alpha(\alpha M_e)^3}{3m_e m_\mu} = 4516.307\ MHz, \quad b_1 = \kappa_\mu \nu_F = 5.266\ MHz, \qquad (11)$$

$$b_2 = \nu_F(1+\kappa_\mu)\left(-3\frac{M_e}{M_\mu} + \frac{2}{3}S_{1/2}\left(\frac{M_e}{M_\mu}\right)^{3/2} + \left(\frac{M_e}{M_\mu}\right)^2 \ln\frac{M_\mu}{M_e} - \frac{7}{64}\left(\frac{M_e}{M_\mu}\right)^2\right) = -64.322\ MHz. \qquad (12)$$

$$c_0 = \nu_F \frac{g_e g_N}{4}\frac{m_\mu}{m_p} = 1083.256\ MHz, \qquad (13)$$

$$c_1 = c_0\left(\frac{3}{2}\frac{M_e}{M_\mu} + \left(\frac{M_e}{M_\mu}\right)^2 \ln\frac{M_\mu}{M_e} + (\ln 2 + \frac{1}{4})\left(\frac{M_e}{M_\mu}\right)^2\right) = 8.323\ MHz. \qquad (14)$$

Note, that, as we determine contributions to the energy spectrum numerically, the corresponding results are presented with an accuracy of 0.001 MHz. We express further the hyperfine splitting contributions in the frequency unit using the relation $\Delta E^{hfs} = 2\pi\hbar\Delta\nu^{hfs}$. Modern numerical values of fundamental physical constants are taken from the paper [25]: the electron mass $m_e = 0.510998910(13)\cdot 10^{-3}\ GeV$, the muon mass $m_\mu = 0.1056583668(38)\ GeV$, the fine structure constant $\alpha^{-1} = 137.035999679(94)$, the helion mass $m(^3_2He) = 2.808391383(70)$ GeV, the muon anomalous magnetic moment $\kappa_\mu = 1.16592069(60)\cdot 10^{-3}$, the electron anomalous magnetic moment $\kappa_e = 1.15965218111(74)\cdot 10^{-3}$.

II. EFFECTS OF THE VACUUM POLARIZATION

The vacuum effects change the interaction (2), (3), (5) between particles in muonic helium atom. One of the most important contributions to HFS is determined by the one-loop vacuum polarization (VP) and electron vertex operator. Indeed, the vacuum loop leads to

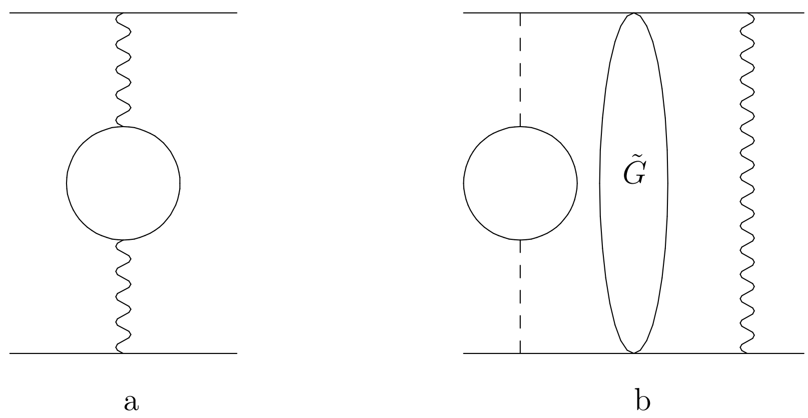


FIG. 1: The vacuum polarization effects. The dashed line represents the Coulomb photon. The wave line represents the hyperfine part of the Breit potential. $\tilde{G}$ is the reduced Coulomb Green's function.

additional factor α/π in the interaction operator, so that corresponding correction to HFS is of the fifth order over fine structure constant. At the same time, the electron vacuum polarization and vertex corrections to the hyperfine splitting of the ground state contain the parameter equal to the ratio of the Compton wave length of the electron and the radius of the Bohr orbit in the subsystem $(\mu^3_2He)^+$: $m_\mu\alpha/m_e$= 1.50886 . . . . It appears in the matrix elements with the use of the bound state wave function in which the characteristic momentum is of order $m_\mu\alpha$. It is impossible to use the expansion over α for such contributions to the energy spectrum. So, we calculate them performing the analytical or numerical integration over the particle coordinates and other parameters without an expansion in α. The effect of the electron vacuum polarization leads to the appearance of a number of additional corrections to the Coulomb potential which we present in the form [13, 26]:

$$\Delta V^{eh}_{VP}(x_e)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)\left(-\frac{2\alpha}{x_e}\right)e^{-2m_e\xi x_e}d\xi,\quad \rho(\xi)=\frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}, \tag{15}$$

$$\Delta V^{\mu h}_{VP}(x_\mu)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)\left(-\frac{2\alpha}{x_\mu}\right)e^{-2m_e\xi x_\mu}d\xi, \tag{16}$$

$$\Delta V^{e\mu}_{VP}(|\mathbf{x}_e-\mathbf{x}_\mu|)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)\frac{\alpha}{x_{e\mu}}e^{-2m_e\xi x_{e\mu}}d\xi, \tag{17}$$

where $x_{e\mu}=|\mathbf{x}_e-\mathbf{x}_\mu|$. They give contributions to the hyperfine splitting in the second order perturbation theory and are discussed below. In the first order perturbation theory the contribution of the vacuum polarization is connected with the modification of the hyperfine splitting part of the Hamiltonian (5) (the diagram (a) in Fig.1). In the coordinate representation it is determined by the integral expression [27–29]:

$$\Delta V^{hfs,e\mu}_{VP}(\mathbf{x}_{e\mu})=-\frac{8\alpha}{3m_em_\mu}(\mathbf{s}_e\cdot\mathbf{s}_\mu)\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\left[\pi\delta(\mathbf{x}_{\mathbf{e}\mu})-\frac{m_e^2\xi^2}{x_{e\mu}}e^{-2m_e\xi x_{e\mu}}\right], \tag{18}$$

$$\Delta V^{hfs,eh}_{VP}(\mathbf{x}_e)=-\frac{8\alpha g_N}{6m_em_p}(\mathbf{s}_e\cdot\mathbf{I}_N)\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\left[\pi\delta(\mathbf{x}_\mathbf{e})-\frac{m_e^2\xi^2}{x_e}e^{-2m_e\xi x_e}\right]. \tag{19}$$

Averaging the potential (18) over the wave function (4) we obtain the following contribution to the hyperfine splitting:

$$b_{\ VP}=\frac{8\alpha^2}{9m_em_\mu}\frac{(\alpha M_e)^3(2\alpha M_\mu)^3}{\pi^3}\int_1^\infty\rho(\xi)d\xi\int d\mathbf{x}_e\int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu}e^{-2\alpha M_e x_e}\times \tag{20}$$

$$\times\left[\pi\delta(\mathbf{x}_\mu-\mathbf{x}_e)-\frac{m_e^2\xi^2}{|\mathbf{x}_\mu-\mathbf{x}_e|}\right]e^{-2m_e\xi|\mathbf{x}_\mu-\mathbf{x}_e|}.$$

There are two integrals over the muon and electron coordinates in Eq.(20) which can be calculated analytically:

$$I_1=\int d\mathbf{x}_e\int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu}e^{-2\alpha M_e x_e}\pi\delta(\mathbf{x}_\mu-\mathbf{x}_e)=\frac{\pi^2}{8\alpha^3M_\mu^3\left(1+\frac{M_e}{2M_\mu}\right)^3}, \tag{21}$$

$$I_2=\int d\mathbf{x}_e\int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu}e^{-2\alpha M_e x_e}\frac{1}{|\mathbf{x}_\mu-\mathbf{x}_e|}e^{-2m_e\xi|\mathbf{x}_\mu-\mathbf{x}_e|}= \tag{22}$$

$$=\frac{32\pi^2}{(4\alpha M_\mu)^5}\frac{\left[\frac{M_e^2}{4M_\mu^2}+\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)^2+\frac{M_e}{2M_\mu}\left(3+\frac{m_e\xi}{M_\mu\alpha}\right)\right]}{\left(1+\frac{M_e}{2M_\mu}\right)^3\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)^2\left(\frac{M_e}{2M_\mu}+\frac{m_e\xi}{2M_\mu\alpha}\right)^2}.$$

They are divergent separately in the subsequent integration over the parameter ξ. But their sum is finite and can be written in the integral form:

$$b_{\ VP}=\nu_F\frac{\alpha M_e}{6\pi M_\mu\left(1+\frac{M_e}{2M_\mu}\right)^3}\int_1^\infty\rho(\xi)d\xi\frac{\left[\frac{M_e}{2M_\mu}+2\frac{m_e\xi}{2M_\mu\alpha}\frac{M_e}{2M_\mu}+\frac{m_e\xi}{2M_\mu\alpha}\left(2+\frac{m_e\xi}{2M_\mu\alpha}\right)\right]}{\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)^2\left(\frac{M_e}{2M_\mu}+\frac{m_e\xi}{2M_\mu\alpha}\right)^2}=0.036\ MHz. \tag{23}$$

The order of this contribution is determined by two small parameters α and M_e/M_μ which are written explicitly. The correction b_{VP} is of the fifth order in α and the first order in the ratio of the electron and muon masses. The contribution of the muon vacuum polarization to the hyperfine splitting is extremely small ($\sim 10^{-6}$ MHz). One should expect that two-loop vacuum polarization contributions to the hyperfine structure are suppressed relative to the one-loop VP contribution by the factor α/π. This means that at present level of accuracy we can neglect these corrections because their numerical value is not exceeding 0.001 MHz. Higher orders of the perturbation theory which contain one-loop vacuum polarization and the Coulomb interaction (3) lead to the recoil corrections of order $\nu_F\alpha\frac{M_e^2}{M_\mu^2}\ln\frac{M_\mu}{M_e}$. Such terms which can contribute 0.004 MHz are included in the theoretical error.

Similar contribution to the coefficient c of order α^6 can be found analytically using the potential (19) ($\alpha_1=\alpha M_e/m_e$):

$$c_{\ VP}=\nu_F\frac{\alpha g_N m_\mu}{6\pi m_p}\frac{\sqrt{1-\alpha_1^2}(6\alpha_1+\alpha_1^3-3\pi)+(6-3\alpha_1^2+6\alpha_1^4)\arccos\alpha_1}{3\alpha_1^3\sqrt{1-\alpha_1^2}}=0.021\ MHz. \tag{24}$$

Let us consider corrections of the electron vacuum polarization (15)-(17) in the second order perturbation theory (SOPT) (the diagram (b) in Fig.1). The contribution of the Coulomb electron-nucleus interaction (15) to the hyperfine splitting can be written as follows:

$$b_{\ VP,\ SOPT,\ e-h}=\frac{16\pi\alpha}{3m_em_\mu}\int d\mathbf{x}_1\int d\mathbf{x}_2\int d\mathbf{x}_3\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\psi^*_{\mu0}(\mathbf{x}_3)\psi^*_{e0}(\mathbf{x}_3)\times \tag{25}$$

$$\times \sum_{n,n'\neq 0}^{\infty} \frac{\psi_{\mu n}(\mathbf{x}_3)\psi_{en'}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2)\psi^*_{en'}(\mathbf{x}_1)}{E_{\mu 0}+E_{e0}-E_{\mu n}-E_{en'}} e^{-2m_e\xi x_1}\psi_{\mu 0}(\mathbf{x}_2)\psi_{e0}(\mathbf{x}_1),$$

where the indices at the coefficient b indicate vacuum polarization contribution (VP) in the second order PT (SOPT) when the electron-helion Coulomb VP potential is considered. The summation in (25) is carried out over the complete system of the eigenstates of the electron and muon excluding the state with $n, n' = 0$. The computation of the expression (25) is simplified with the use of the orthogonality condition for the muon wave functions:

$$b_{\ VP,\ SOPT,\ e-h} = \nu_F \frac{32\alpha M_e^2}{3\pi M_\mu^2}\int_1^\infty \rho(\xi)d\xi \int_0^\infty x_2^3 dx_3 \int_0^\infty x_1 dx_1 e^{-x_1\frac{M_e}{M_\mu}\left(1+\frac{m_e\xi}{\alpha M_e}\right)} e^{-2x_3\left(1+\frac{M_e}{2M_\mu}\right)} \times \tag{26}$$

$$\left[\frac{M_\mu}{M_e x_>} - \ln\left(\frac{M_e}{M_\mu}x_<\right) - \ln\left(\frac{M_e}{M_\mu}x_>\right) + Ei\left(\frac{M_e}{M_\mu}x_<\right) + \frac{7}{2} - 2C - \frac{M_e}{2M_\mu}(x_1+x_3) + \frac{1-e^{\frac{M_e}{M_\mu}x_<}}{\frac{M_e}{M_\mu}x_<}\right] =$$

$$= 0.150\ MHz,$$

where $x_< = \min(x_1, x_3)$, $x_> = \max(x_1, x_3)$, $C = 0.577216\ldots$ is the Euler's constant and $Ei(x)$ is the exponential-integral function. It is necessary to emphasize that the transformation of the expression (25) into (26) is carried out with the use of the compact representation for the electron reduced Coulomb Green's function obtained in Refs.[30]:

$$G_e(\mathbf{x}_1, \mathbf{x}_3) = \sum_{n\neq 0}^{\infty} \frac{\psi_{en}(\mathbf{x}_3)\psi^*_{en}(\mathbf{x}_1)}{E_{e0}-E_{en}} = -\frac{\alpha M_e^2}{\pi} e^{-\alpha M_e(x_1+x_3)}\left[\frac{1}{2\alpha M_e x_>} - \right. \tag{27}$$

$$\left. -\ln(2\alpha M_e x_>) - \ln(2\alpha M_e x_<) + Ei(2\alpha M_e x_<) + \frac{7}{2} - 2C - \alpha M_e(x_1+x_3) + \frac{1-e^{2\alpha M_e x_<}}{2\alpha M_e x_<}\right].$$

The contribution (26) has the same order of the magnitude $O(\alpha^5\frac{M_e}{M_\mu})$ as the previous correction (23) in the first order perturbation theory. Similar calculation can be performed in the case of muon-nucleus Coulomb vacuum polarization potential (16). The intermediate electron state is the 1S state and the reduced Coulomb Green's function of the system appearing in the second order PT transforms to the Green's function of the muon. The correction of the operator (16) to the hyperfine splitting (coefficient b) is obtained in the following integral form:

$$b_{\ VP,\ SOPT,\ \mu-h} = \nu_F \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi \int_0^\infty x_3^2 dx_3 \int_0^\infty x_2 dx_2 e^{-x_3\left(1+\frac{M_e}{2M_\mu}\right)} e^{-x_2\left(1+\frac{m_e\xi}{2M_\mu\alpha}\right)} \times \tag{28}$$

$$\times\left[\frac{1}{x_>} - \ln x_> - \ln x_< + Ei(x_<) + \frac{7}{2} - 2C - \frac{x_2+x_3}{2} + \frac{1-e^{x_<}}{x_<}\right] = 0.048\ MHz.$$

The vacuum polarization correction to HFS which is determined by the operator (17) in the second order perturbation theory is the most difficult for the calculation. Indeed, in this case we have to consider the intermediate excited states both for the muon and electron. Following Ref.[22] we have divided total contribution into two parts. The first part in which the intermediate muon is in the 1S state can be written as:

$$b_{\ VP,\ SOPT,\ \mu-e}(n=0) = \frac{256\alpha^2(\alpha M_e)^3(2\alpha M_\mu)^3}{9}\int_0^\infty x_3^2 dx_3 \times \tag{29}$$

$$\times\int_0^\infty x_1^2dx_1e^{-\alpha(M_e+4M_\mu)x_3}\int_1^\infty\rho(\xi)d\xi\Delta V_{VP\ \mu}(x_1)G_e(x_1,x_3),$$

where the function $V_{VP\ \mu}(x_1)$ is equal

$$\Delta V_{VP\ \mu}(x_1)=\int d\mathbf{x}_2e^{-4\alpha M_\mu x_2}\frac{(2\alpha M_\mu)^3}{\pi}\frac{\alpha}{|\mathbf{x}_1-\mathbf{x}_2|}e^{-2m_e\xi|\mathbf{x}_1-\mathbf{x}_2|}= \tag{30}$$

$$=\frac{32\alpha^4M_\mu^3}{x_1(16\alpha^2M_\mu^2-4m_e^2\xi^2)^2}\left[8\alpha M_\mu\left(e^{-2m_e\xi x_1}-e^{-4\alpha M_\mu x_1}\right)+x_1(4m_e^2\xi^2-16\alpha^2M_\mu^2)e^{-4\alpha M_\mu x_1}\right].$$

After the substitution (30) in (29) the numerical integration gives the result:

$$b_{\ VP,\ SOPT}(n=0)=-0.029\ MHz. \tag{31}$$

Second part of the vacuum polarization correction to the hyperfine splitting due to the electron-muon interaction (17) can be presented as follows:

$$b_{\ VP,\ SOPT,\ \mu e}(n\neq0)=-\frac{16\alpha^2}{9m_em_\mu}\int d\mathbf{x}_3\int d\mathbf{x}_2\int_1^\infty\rho(\xi)d\xi\psi^*_{\mu0}(\mathbf{x}_3)\psi^*_{e0}(\mathbf{x}_3)\times \tag{32}$$

$$\times\sum_{n\neq0}\psi_{\mu n}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2)\frac{M_e}{2\pi}\frac{e^{-\mathcal{B}|\mathbf{x}_3-\mathbf{x}_1|}}{|\mathbf{x}_3-\mathbf{x}_1|}\frac{\alpha}{|\mathbf{x}_2-\mathbf{x}_1|}e^{-2m_e\xi|\mathbf{x}_2-\mathbf{x}_1|}\psi_{\mu0}(\mathbf{x}_2)\psi_{e0}(\mathbf{x}_1).$$

In the expression (32) we have replaced the exact electron Coulomb Green's function by the free electron Green's function which contains $\mathcal{B}=[2M_e(E_{\mu n}-E_{\mu0}-E_{e0}]^{1/2}$. (see more detailed discussion of this approximation in Refs.[22, 23]). We also replace the electron wave functions by their values at the origin as in Ref.[22] neglecting higher order recoil corrections. After that the integration over $\mathbf{x}_1$ can be done analytically:

$$J=\int d\mathbf{x}_1\frac{e^{-\mathcal{B}|\mathbf{x}_3-\mathbf{x}_1|}}{|\mathbf{x}_3-\mathbf{x}_1|}\frac{e^{-2m_e\xi|\mathbf{x}_2-\mathbf{x}_1|}}{|\mathbf{x}_2-\mathbf{x}_1|}=-\frac{4\pi}{|\mathbf{x}_3-\mathbf{x}_2|}\frac{1}{\mathcal{B}^2-4m_e^2\xi^2}\left[e^{-\mathcal{B}|\mathbf{x}_3-\mathbf{x}_2|}-e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_2|}\right]= \tag{33}$$

$$=2\pi\Bigg[\frac{\left(1-e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_2|}\right)}{2m_e^2\xi^2|\mathbf{x}_3-\mathbf{x}_2|}-\frac{\mathcal{B}}{2m_e^2\xi^2}+\frac{\left(1-e^{-2m_e\xi|\mathbf{x}_3-\mathbf{x}_2|}\right)\mathcal{B}^2}{8m_e^4\xi^4|\mathbf{x}_3-\mathbf{x}_2|}+\frac{\mathcal{B}^2|\mathbf{x}_3-\mathbf{x}_2|}{4m_e^2\xi^2}-$$

$$-\frac{\mathcal{B}^3}{8m_e^4\xi^4}-\frac{\mathcal{B}^3(\mathbf{x}_3-\mathbf{x}_1)^2}{12m_e^2\xi^2}+...\Bigg],$$

where we have performed the expansion of the first exponential in the square brackets over powers of $\mathcal{B}|\mathbf{x}_3-\mathbf{x}_2|$. As discussed in Ref.[22] one can treat this series as an expansion over the recoil parameter $\sqrt{M_e/M_\mu}$. For the further transformation the completeness condition is useful:

$$\sum_{n\neq0}\psi_{\mu n}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2)=\delta(\mathbf{x}_3-\mathbf{x}_2)-\psi_{\mu0}(\mathbf{x}_3)\psi^*_{\mu0}(\mathbf{x}_2). \tag{34}$$

The wave function orthogonality leads to the zero results for the second and fifth terms in the square brackets of Eq.(33). The first term in Eq.(33) gives the leading order contribution in two small parameters α and M_e/M_μ:

$$b_{\ VP,\ SOPT,\ \mu-e}(n\neq0)=b_{11}+b_{12},\ \ b_{11}=-\frac{3\alpha^2M_e}{8m_e}\nu_F, \tag{35}$$

$$b_{12}=\nu_F\frac{2\alpha^2}{3\pi\frac{m_e}{M_e}}\int_1^\infty\rho(\xi)\frac{d\xi}{\xi}\frac{M_\mu^4\alpha^4}{(4\alpha M_\mu+2m_e\xi)^4}\left[256+232\frac{m_e\xi}{M_\mu\alpha}+80\frac{m_e^2\xi^2}{M_\mu^2\alpha^2}+10\frac{m_e^3\xi^3}{M_\mu^3\alpha^3}\right]. \quad (36)$$

The numerical value of the sum $b_{11}+b_{12}=-0.062$ MHz is included in Table I. It is important to calculate also the contributions of other terms of the expression (33) to the hyperfine splitting. Taking the fourth term in Eq.(33) which is proportional to $\mathcal{B}^2=2M_e(E_{\mu n}-E_{\mu 0})$ we have performed the sequence of the transformations in the coordinate representation:

$$\sum_{n=0}^{\infty}E_{\mu n}\int d\mathbf{x}_2\int d\mathbf{x}_3\psi^*_{\mu 0}(\mathbf{x}_2)\psi_{\mu n}(\mathbf{x}_3)\psi^*_{\mu n}(\mathbf{x}_2)|\mathbf{x}_3-\mathbf{x}_2|\psi_{\mu 0}(\mathbf{x}_2)= \quad (37)$$

$$=\int d\mathbf{x}_2\int d\mathbf{x}_3\delta(\mathbf{x}_3-\mathbf{x}_2)\left[-\frac{\nabla_3^2}{2M_\mu}|\mathbf{x}_3-\mathbf{x}_2|\psi^*_{\mu 0}(\mathbf{x}_3)\right]\psi_{\mu 0}(\mathbf{x}_2).$$

Evidently, we have the divergent expression in Eq.(37) due to the presence of the δ-function. The same divergence occurs in the other term containing $\mathcal{B}^2$ in the square brackets of Eq.(33). But their sum is finite and can be calculated analytically with the result:

$$b_{\mathcal{B}^2}=\nu_F\frac{\alpha^2M_e^2}{m_eM_\mu}\left(18-5\frac{\alpha^2M_\mu^2}{m_e^2}\right). \quad (38)$$

Numerical value of this correction 0.009 MHz is smaller than the leading order term. Let us consider also the nonzero term in Eq.(33) proportional to $\mathcal{B}^3$. First of all, it can be transformed to the following expression after the integration over ξ:

$$b_{\mathcal{B}^3}=-\nu_F\frac{4\alpha^3}{45\pi}\sqrt{\frac{M_e}{M_\mu}}\frac{M_e^2}{m_e^2}S_{3/2}, \quad (39)$$

where the sum $S_{3/2}$ is defined as follows:

$$S_p=\sum_n\left[\left(\frac{E_{\mu n}-E_{\mu 0}}{R_\mu}\right)^p\right]|<\psi_{\mu 0}|\frac{\mathbf{x}}{a_\mu}|\psi_{\mu n}>|^2, \quad (40)$$

$R_\mu=2\alpha^2M_\mu$, $a_\mu=1/2\alpha M_\mu$. Using the known analytical expressions for the dipole matrix elements entering in Eq.(40) in the case of the discrete and continuous spectrum [12, 31] we can write separately their contributions to the sum $S_{3/2}$ in the form:

$$S^d_{3/2}=\sum_{n=0}^{\infty}\frac{2^8n^4(n-1)^{2n-\frac{7}{2}}}{(n+1)^{2n+\frac{7}{2}}}=1.50989\ldots, \quad (41)$$

$$S^c_{3/2}=\int_0^\infty kdk\frac{2^8}{\left(1-e^{-\frac{2\pi}{k}}\right)}\frac{1}{(1+k^2)^{\frac{7}{2}}}|\left(\frac{1+ik}{1-ik}\right)^{\frac{i}{k}}|^2=1.76236\ldots. \quad (42)$$

As a result $S_{3/2}=3.2722\ldots$. The similar calculation of the sum $S_{1/2}$ relating to this problem (see Ref.[22]) gives $S_{1/2}=2.9380\ldots$. Numerical value (39) is taken into account in the total result presented in Table I.

The Coulomb vacuum polarization (15) does not contain the muon coordinate, so, its contribution to the coefficient c in the second order PT can be derived taking $n=0$ for the

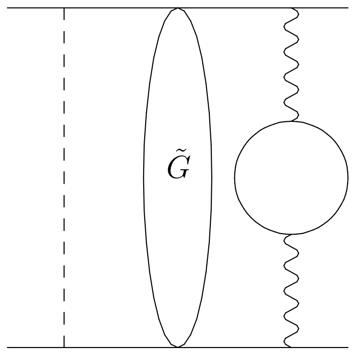


FIG. 2: Vacuum polarization effects in the second order perturbation theory. The dashed line represents the first part of the potential ΔH (3). The wave line represents the hyperfine part of the Breit potential.

muon state in the Coulomb Green's function. Moreover, the $\delta(\mathbf{x}_e)$ function in (5) results in the appearance of the electron Green's function with one zero argument:

$$G_e(\mathbf{x}) = \sum_{n\neq 0} \frac{\psi_{e\ n}(0)\psi^+_{e\ n}(\mathbf{x})}{E_{e\ 0}-E_{e\ n}} = \frac{M_e e^{-\alpha M_e x}}{4\pi x}\left[4\alpha M_e x(\ln(2\alpha M_e x)+C)+4\alpha^2 M_e^2 x^2 - 10\alpha M_e x - 2\right]. \quad (43)$$

Corresponding value of the hyperfine splitting is equal

$$c_{\ VP,\ SOPT,\ e-h} = \nu_F \frac{\alpha m_\mu g_N}{2\pi m_p}\int_1^\infty \rho(\xi)d\xi \frac{2-(1+\frac{\xi}{\alpha_1})(3+2(1+\frac{\xi}{\alpha_1}))-2(1+\frac{\xi}{\alpha_1})\ln(1+\frac{\xi}{\alpha_1})}{(1+\frac{\xi}{\alpha_1})^3} = \quad (44)$$

$$= -0.044\ MHz.$$

The vacuum polarization in the Coulomb $(\mu - N)$ interaction does not contribute to c in SOPT because of the orthogonality of the muon wave functions. Let consider correction to the coefficient c arose from (17) in SOPT. Only intermediate muon state with $n=0$ in the Green's function gives the contribution in this case. Using (43) we make integration over electron coordinates and present this correction in the form ($\gamma = m_e\xi/2\alpha M_\mu$, $\gamma_1 = M_e/4M_\mu$):

$$c_{\ VP,\ SOPT,\ e-\mu} = -\nu_F \frac{\alpha m_\mu g_N M_e}{24\pi m_p M_\mu}\int_1^\infty \rho(\xi)d\xi \frac{2(1-\gamma)^2}{(1-\gamma^2)^2(1+2\gamma_1)^4(\gamma+2\gamma_1)^3}\times \quad (45)$$

$$[-\gamma^3(1+2\gamma_1(7+6\gamma_1))-2\gamma^2(1+3\gamma_1)(1+2\gamma_1(7+6\gamma_1))-4\gamma_1^2[5+2\gamma_1(17+12\gamma_1(2+\gamma_1))]-2\gamma\gamma_1\times$$
$$[9+2\gamma_1(37+4\gamma_1(17+9\gamma_1))]+4\gamma_1(1+2\gamma_1)(\gamma+2\gamma_1)(1+\gamma^2+2\gamma(1+2\gamma_1)+2\gamma_1(3+2\gamma_1))\ln(2\gamma_1)] =$$
$$= 0.009\ MHz.$$

There exist another contributions of the second order perturbation theory in which we have the vacuum polarization perturbation connected with the hyperfine splitting parts of the Breit potential (18), (19) (see Fig.2). Other perturbation potential in this case is determined by the first term of relation (3). We can divide the HFS correction of (18) into two parts. One part with $n=0$ corresponds to the ground state muon. The other part with

$n \neq 0$ accounts the excited muon states. The δ-function term in Eq.(18) gives the following contribution to HFS at $n = 0$ (compare with Ref.[22]):

$$b_{\ VP,\ SOPT,\ 11}(n=0) = \nu_F \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \frac{11 M_e}{16 M_\mu}. \tag{46}$$

Obviously, this integral in the variable ξ is divergent. So, we have to consider the contribution of the second term of the potential (18) to the hyperfine splitting which is determined by the more complicated expression:

$$b_{\ VP,\ SOPT,\ 12}(n=0) = \frac{16\alpha^2 m_e^2}{9\pi m_e m_\mu} \int_1^\infty \rho(\xi)\xi^2 d\xi \int d\mathbf{x}_3 \psi_{e0}(\mathbf{x}_3) \Delta V_1(\mathbf{x}_3) \times \tag{47}$$

$$\times \sum_{n' \neq 0} \frac{\psi_{en'}(\mathbf{x}_3)\psi^*_{en'}(\mathbf{x}_1)}{E_{e0} - E_{en'}} \Delta V_2(\mathbf{x}_1) \psi_{e0}(\mathbf{x}_1),$$

where

$$\Delta V_1(\mathbf{x}_3) = \int d\mathbf{x}_4 \psi^*_{\mu 0}(\mathbf{x}_4) \frac{e^{-2m_e\xi|\mathbf{x}_3 - \mathbf{x}_4|}}{|\mathbf{x}_3 - \mathbf{x}_4|} \psi_{\mu 0}(\mathbf{x}_4) = \tag{48}$$

$$= \frac{4(2\alpha M_\mu)^3}{x_3[(4\alpha M_\mu)^2 - (2m_e\xi)^2]^2} \left[8\alpha M_\mu e^{-2m_e \xi x_3} + e^{-4\alpha M_\mu x_3} \left(-8\alpha M_\mu - 16\alpha^2 M_\mu^2 x_3 + 4m_e^2 \xi^2 x_3 \right) \right],$$

$$\Delta V_2(\mathbf{x}_1) = \int d\mathbf{x}_2 \psi_{\mu 0}(\mathbf{x}_2) \left(\frac{\alpha}{|\mathbf{x}_2 - \mathbf{x}_1|} - \frac{\alpha}{x_1} \right) \psi_{\mu 0}(\mathbf{x}_2) = -\frac{\alpha}{x_1}(1 + 2\alpha M_\mu x_1) e^{-4\alpha M_\mu x_1}. \tag{49}$$

Nevertheless, integrating in Eq.(47) over all coordinates we obtain the following result in the leading order in the ratio (M_e/M_μ):

$$b_{\ VP,\ SOPT,\ 12}(n=0) = -\nu_F \frac{m_e}{M_e} \frac{M_e^2}{96\pi M_\mu^2} \int_1^\infty \rho(\xi)\xi d\xi \frac{32 + 63\gamma + 44\gamma^2 + 11\gamma^3}{(1+\gamma)^4}. \tag{50}$$

This integral also has the divergence at large values of the parameter ξ. But the sum of integrals (46) and (50) is finite:

$$b_{\ VP,\ SOPT,\ 11}(n=0) + b_{\ VP,\ SOPT,\ 12}(n=0) = \nu_F \frac{\alpha M_e}{48\pi M_\mu} \int_1^\infty \rho(\xi) d\xi \frac{11 + 12\gamma + 3\gamma^2}{(1+\gamma)^4} = 0.008\ MHz. \tag{51}$$

Let us consider now the terms in the coefficient b with $n \neq 0$. The delta-like term of the potential (18) gives the contribution to the HFS known from the calculation of Ref.[22]:

$$b_{\ VP,\ SOPT,\ 21}(n \neq 0) = \nu_F \frac{\alpha}{3\pi M_\mu} \int_1^\infty \rho(\xi) d\xi \left(-\frac{35 M_e}{16 M_\mu} \right). \tag{52}$$

Another correction from the second term of the expression (18) can be simplified after the replacement the exact electron Green's function by the free electron Green's function:

$$b_{\ VP,\ SOPT,\ 22}(n \neq 0) = -\frac{16\alpha^3 M_e m_e^2}{9 m_e m_\mu} \int_1^\infty \rho(\xi)\xi^2 d\xi \int d\mathbf{x}_2 \int d\mathbf{x}_3 \times \tag{53}$$

$$\times \int d\mathbf{x}_4 \psi^*_{\mu 0}(\mathbf{x}_4) \frac{e^{-2m_e\xi|\mathbf{x}_3 - \mathbf{x}_4|}}{|\mathbf{x}_3 - \mathbf{x}_4|} \sum_{n \neq 0}^{\infty} \psi_{\mu n}(\mathbf{x}_4) \psi_{\mu n}(\mathbf{x}_2) |\mathbf{x}_3 - \mathbf{x}_2| \psi_{\mu 0}(\mathbf{x}_2)$$

The analytical integration in Eq.(53) over all coordinates leads to the result:

$$b_{\ VP,\ SOPT,\ 22}(n\neq 0)=-\nu_F\frac{\alpha M_e}{3\pi M_\mu}\int_1^\infty\rho(\xi)d\xi\left[\frac{1}{\gamma}-\frac{1}{(1+\gamma)^4}\left(4+\frac{1}{\gamma}+10\gamma+\frac{215\gamma^2}{16}+\frac{35\gamma^4}{16}\right)\right]. \tag{54}$$

The sum of expressions (52) and (54) gives again the finite contribution to the hyperfine splitting:

$$b_{\ VP,\ SOPT,\ 21}(n\neq 0)+\Delta\nu^{hfs}_{VP\ SOPT\ 22}(n\neq 0)= \tag{55}$$

$$=-\nu_F\frac{\alpha M_e}{3\pi M_\mu}\int_1^\infty\rho(\xi)d\xi\frac{35+76\gamma+59\gamma^2+16\gamma^3}{16(1+\gamma)^4}=-0.062\ MHz.$$

Despite the fact that the absolute values of the calculated VP corrections (31), (35), (36), (37), (39), (51), (55) are sufficiently large , their summary contribution to the hyperfine splitting (see Table I) is small because they have different signs.

The hyperfine splitting interaction (19) gives the contributions to the coefficient c in second order PT. Since the muon coordinate does not enter into the expression (19), we should set $n=0$ for the muon intermediate states in the Green's function. The initial formula for this correction is

$$c_{\ VP,\ SOPT}=\frac{8\alpha^3 g_N}{9\pi m_e m_p}\int_1^\infty\rho(\xi)d\xi\int d\mathbf{x}_1\int d\mathbf{x}_3\int d\mathbf{x}_4|\psi_{\mu 0}(\mathbf{x}_3)|^2\psi^*_{e0}(\mathbf{x}_4)\psi_{e0}(\mathbf{x}_1)\times \tag{56}$$

$$\times\left[\frac{1}{|\mathbf{x}_3-\mathbf{x}_4|}-\frac{1}{x_4}\right]G_e(\mathbf{x}_4,\mathbf{x}_1)\left(\pi\delta(\mathbf{x}_1)-\frac{m_e^2\xi^2}{x_1}e^{-2m_e\xi x_1}\right).$$

The integration over $\mathbf{x}_3$ can be done analytically as in (49). Then it is useful to divide (56) into two parts. The coordinate integration in the first term with the δ - function is performed by means of (43). In the second term of (56) we use the electron Green's function in the form (27). The summary result can be presented in the integral form in the leading order in M_e/M_μ:

$$c_{\ VP,\ SOPT}=\frac{2\alpha^5 g_N M_e^4}{9\pi m_e m_p M_\mu}\int_1^\infty\rho(\xi)d\xi\frac{3+2\frac{m_e\xi}{2\alpha M_\mu}}{(1+\frac{m_e\xi}{2\alpha M_\mu})^2}=0.013\ MHz. \tag{57}$$

III. NUCLEAR STRUCTURE AND RECOIL EFFECTS

Another set of significant corrections to the hyperfine splitting of muonic helium atom which we study in this work is determined by the nuclear structure and recoil [14, 32–34]. The charge and magnetic moment distributions of the helion are described by two form factors $G_E(k^2)$ and $G_M(k^2)$ for which we use the known parameterization [35]:

$$G_E(k^2)=e^{-\tilde a^2k^2}-\tilde b^2k^2e^{-\tilde c^2k^2}+\tilde d\left[e^{-\frac{(k+\tilde q_0)^2}{\tilde p^2}}+e^{-\frac{(k-\tilde q_0)^2}{\tilde p^2}}\right], \tag{58}$$

$$G_M(k^2)=\left[e^{-\tilde a_1^2k^2}-\tilde b_1^2k^2e^{-\tilde c_1^2k^2}\right], \tag{59}$$

where numerous parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{a}_1$, $\tilde{b}_1$, $\tilde{c}_1$, $\tilde{q}_0$, $\tilde{p}$ are written explicitly in [35]. In 1γ - interaction the nuclear structure correction to the coefficient c is determined by the amplitudes shown in Fig.3. Purely point contribution in Fig.3(b) leads to the HFS value (13). Then the nuclear structure correction is given by

$$c_{\ str,\ 1\gamma} = \nu_F \frac{(1+\kappa_e) g_N m_\mu}{2m_p} \left[\int G_M(x) e^{-2\alpha M_e x} d\mathbf{x} - 1 \right] = -0.072\ MHz. \tag{60}$$

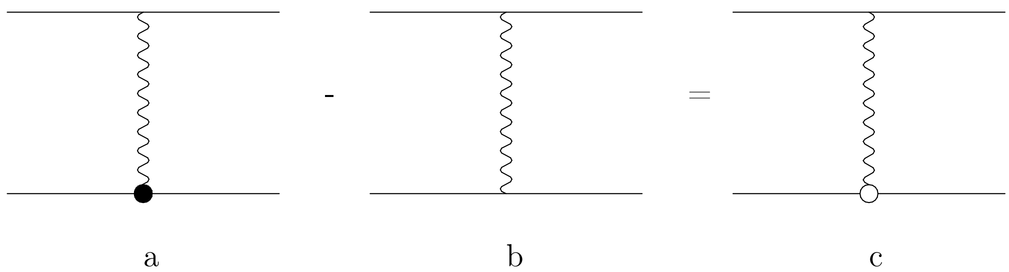


FIG. 3: Nuclear structure correction to coefficient c in 1γ interaction. The bold point represents the nuclear vertex operator. The wave line represents the hyperfine part of the Breit potential.

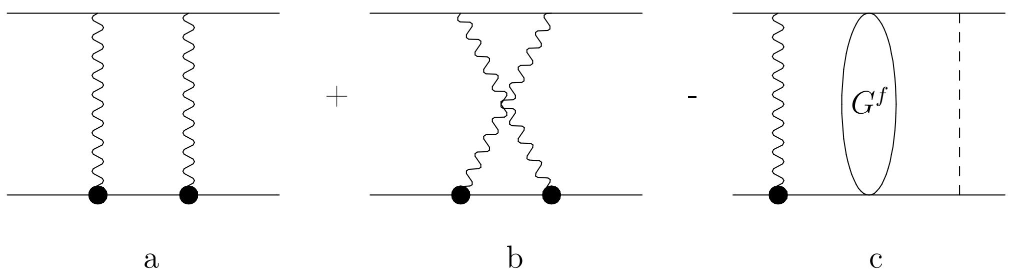


FIG. 4: Nuclear structure corrections to coefficient c in 2γ interactions. The bold point represents the nuclear vertex operator. The wave line represents the hyperfine part of the Breit potential. Dash line corresponds to the Coulomb potential.

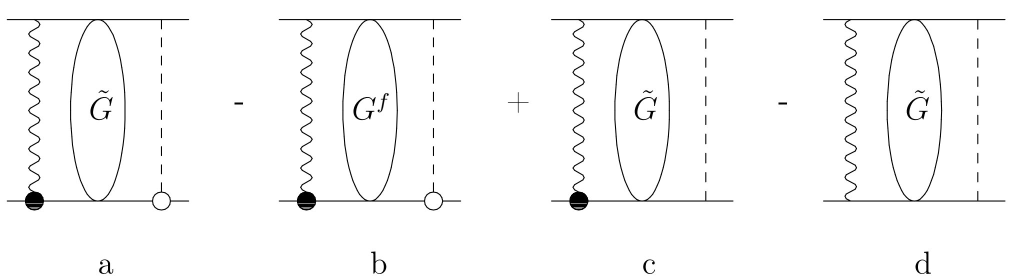


FIG. 5: Nuclear structure corrections to coefficient c in second order PT. The bold point represents the nuclear vertex operator. The wave line represents the hyperfine part of the Breit potential. Dash line corresponds to the Coulomb potential. $\tilde{G}$ is the reduced Coulomb Green's function.

Two-photon $(e-h)$ interaction amplitudes (see Fig.4) give the contribution to HFS of order α^5. It can be presented in terms of the Dirac and Pauli form factors F_1 and F_2 [17, 28]:

$$c_{\ str,\ 2\gamma} = \frac{Z^2\alpha^5 M_e^3}{3\pi m_e m_h} \delta_{l0} \int_0^\infty \frac{dk}{k} V(k), \tag{61}$$

$$V(k)=\frac{2F_2^2k^2}{m_em_h}+\frac{M_e}{(m_e-m_h)k(k+\sqrt{4m_e^2+k^2})}\Big[-128F_1^2m_e^2-128F_1F_2m_e^2+16F_1^2k^2+$$

$$+64F_1F_2k^2+16F_2^2k^2+\frac{32F_2^2m_e^2k^2}{m_h^2}+\frac{4F_2^2k^4}{m_e^2}-\frac{4F_2^2k^4}{m_h^2}\Big]+\frac{M_e}{(m_e-m_h)k(k+\sqrt{4m_h^2+k^2})}\times$$

$$\times\left[128F_1^2m_h^2+128F_1F_2m_h^2-16F_1^2k^2-64F_1F_2k^2-48F_2^2k^2\right].$$

The subtraction term in Fig.4 is taken as follows:

$$c_{\ iter,\ str}=\frac{64}{3}\frac{M_e^4Z^2\alpha^5g_N}{6m_em_h\pi}\int_0^\infty\frac{dk}{k^2}G_M(k^2). \quad (62)$$

It is only the part of the iteration contribution $< V_{1\gamma}\times G^f\times V_{1\gamma}>^{hfs}_{str}$. Other parts are used in the second order PT (see Fig.5). As a result we obtain:

$$c_{\ str,\ 2\gamma}+c_{\ iter,\ str}=-0.077\ MHz. \quad (63)$$

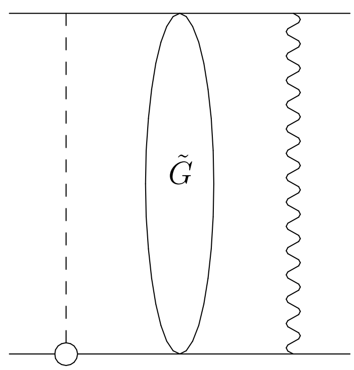


FIG. 6: Nuclear structure correction to coefficient b in the second order perturbation theory. The wave line represents the hyperfine $(e-\mu)$ interaction. $\tilde G$ is the reduced Coulomb Green's function.

The nuclear structure corrections to the coefficient c in second order PT which are presented in Fig.5. Here we have two different contributions. First contribution is related with diagrams in Fig.5(a),(b) when the hyperfine part of one perturbation is determined by magnetic form factor G_M and the other perturbation is described by the charge radius of the helion r_h:

$$\Delta V^C_{str,e-h}(\mathbf{r})=\frac{2}{3}\pi Z\alpha<r_h^2>\delta(\mathbf{r}). \quad (64)$$

The general integral structure of this correction and its numerical value are the following:

$$c_{1,\ str,\ SOPT,e-h}=\nu_F\frac{4\pi Zg_Nm_\mu}{6\alpha M_em_p}<r_h^2>\int_0^\infty xdxe^{-2x}[4x(\ln 2x+C)+4x^2-10x]G_M(\frac{x}{\alpha M_e})= \quad (65)$$

$$=0.00005\ MHz.$$

Numerical value of the contribution $c_{1,\ str,\ SOPT,e-h}$ is obtained by means of the charge radius of the helion $r_h=1.9642(11)$ fm [36]. The second nuclear structure contribution from the diagrams Fig.5 (c), (d) is evaluated by means of the potential ΔH (3) and the

helion magnetic form factor. For the amplitude in Fig.5(c) we make the integration over the muon coordinate in the muon state at $n = 0$ and present the correction to the coefficient c in the form:

$$c_{2,\ str,\ SOPT,e-h}+c_1 = \frac{\pi\alpha M_e^5 g_N(1+\kappa_e)}{24 m_e m_p M_\mu^5}\int_0^\infty x^2 e^{-\frac{M_e}{2M_\mu}x} G_M(\frac{x}{4\alpha M_\mu})dx \int_0^\infty y(1+\frac{y}{2})e^{-y(1+\frac{M_e}{2M_\mu})}dy\times \tag{66}$$

$$\left[\frac{1}{\frac{M_e}{2M_\mu}x_>} - \ln(\frac{M_e}{2M_\mu}x_>) - \ln(\frac{M_e}{2M_\mu}x_<) + Ei(\frac{M_e}{2M_\mu}x_<) + \frac{7}{2} - 2C - \frac{M_e}{2M_\mu}(x+y) + \frac{1-e^{\frac{M_e}{2M_\mu}x_<}}{\frac{M_e}{2M_\mu}x_<}\right] =$$

$$= 8.249\ MHz.$$

Subtracting the point contribution c_1 (14) we find $c_{2,\ str,\ SOPT,e-h} = -0.074$ MHz.

There is the nuclear structure contribution to the coefficient b in second order PT which is presented in Fig.6. If we consider the Coulomb interaction between the muon and helion, then the structure correction takes on form:

$$b_{str,\mu-h} = \frac{64\pi^2\alpha^2}{9m_e m_\mu} r_h^2 \frac{1}{\sqrt{\pi}}\left(2\alpha M_\mu\right)^{3/2}\int d\mathbf{x}_3 \psi^*_{\mu 0}(\mathbf{x}_3)|\psi_{e0}(\mathbf{x}_3)|^2 G_\mu(\mathbf{x}_3, 0, E_{\mu 0}). \tag{67}$$

After that the analytical integration over the coordinate $\mathbf{x}_3$ in Eq.(67) can be carried out using the representation of the muon Green's function similar to expression (43). The result of the integration of order $O(\alpha^6)$ is written as an expansion in the ratio M_e/M_μ:

$$b_{str,\mu-h} = -\nu_F \frac{8}{3}\alpha^2 M_\mu^2 r_\alpha^2\left(3\frac{M_e}{M_\mu} - \frac{11}{2}\frac{M_e^2}{M_\mu^2} + \ldots\right) = -0.010\ MHz. \tag{68}$$

The same approach can be used in the calculation of the electron-nucleus interaction. The electron feels as well the distribution of the helion electric charge. The corresponding contribution of the nuclear structure effect to the hyperfine splitting is determined by the expression:

$$b_{str,e-h} = \frac{64\pi^2\alpha^2}{9m_e m_\mu} r_h^2 \int d\mathbf{x}_1 \int d\mathbf{x}_3 |\psi^*_{\mu 0}(\mathbf{x}_3)|^2 \psi_{e0}(\mathbf{x}_3) G_\mu(\mathbf{x}_3, \mathbf{x}_1, E_{e0})\psi_{e0}(\mathbf{x}_1)\delta(\mathbf{x}_1). \tag{69}$$

Performing the analytical integration in Eq.(69) we obtain the following series:

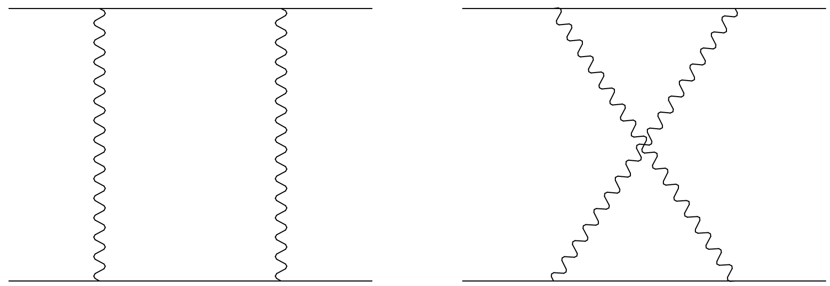
$$b_{str,e-h} = -\nu_F \frac{4}{3}\alpha^2 M_e^2 r_\alpha^2\left[5 - \ln\frac{M_e}{M_\mu} + \frac{M_e^2}{M_\mu^2}\left(3\ln\frac{M_e}{M_\mu} - 7\right) + \frac{M_e^2}{M_\mu^2}\left(\frac{17}{2} - 3\ln\frac{M_e}{M_\mu}\right)\ldots\right] = \tag{70}$$

$$= -0.003\ MHz.$$

We have included in Table I the total nuclear structure contribution to the coefficient b which is equal to the sum of the numerical values (68) and (70).

Special attention has to be given to the recoil corrections connected with two-photon exchange diagrams shown in Fig.7 in the case of the electron-muon interaction. The leading order recoil contribution to the interaction operator between the muon and electron is determined by the expression [5, 13, 37]:

$$\Delta V^{hfs}_{rec,\mu-e}(\mathbf{x}_{\mu e}) = -8\frac{\alpha^2}{m_\mu^2 - m_e^2}\ln\frac{m_\mu}{m_e}(\mathbf{s}_\mu \mathbf{s}_e)\delta(\mathbf{x}_{\mu e}). \tag{71}$$



a b

FIG. 7: Two photon exchange amplitudes in the electron-muon hyperfine interaction.

Averaging the potential $\Delta V^{hfs}_{rec,\mu-e}$ over the wave functions (4) we obtain the recoil correction to the coefficient b:

$$b_{rec,\mu-e}=\nu_F\frac{3\alpha}{\pi}\frac{m_e m_\mu}{m_\mu^2-m_e^2}\ln\frac{m_\mu}{m_e}=0.812\ MHz. \tag{72}$$

There exist also the two-photon interactions between the bound particles of muonic helium atom when one hyperfine photon transfers the interaction from the electron to muon and another Coulomb photon from the electron to the nucleus (or from the muon to the nucleus). Supposing that these amplitudes give smaller contribution to the hyperfine splitting we included them in the theoretical error.

IV. ELECTRON VERTEX CORRECTIONS

In the initial approximation the potential of the hyperfine splitting is determined by Eq.(5). It leads to the energy splitting of order α^4. In QED perturbation theory there is the electron vertex correction to the potential (5) which is defined by the diagram in Fig.8(a). In momentum representation the corresponding operator of hyperfine interaction has the form:

$$\Delta V^{hfs}_{vertex}(k^2)=-\frac{8\alpha^2}{3m_e m_\mu}\left(\frac{\boldsymbol{\sigma}_e\boldsymbol{\sigma}_\mu}{4}\right)\left[G_M^{(e)}(k^2)-1\right], \tag{73}$$

where $G_M^{(e)}(k^2)$ is the electron magnetic form factor. We extracted for the convenience the factor α/π from $\left[G_M^{(e)}(k^2)-1\right]$. Usually used approximation for the electron magnetic form factor $G_M^{(e)}(k^2)\approx G_M^{(e)}(0)=1+\kappa_e$ is not quite correct in this task. Indeed, characteristic momentum of the exchanged photon is $k\sim\alpha M_\mu$. It is impossible to neglect it in the magnetic form factor as compared with the electron mass m_e. So, we should use exact one-loop expression for the magnetic form factor which was obtained by many authors [26]. Let us note that the Dirac form factor of the electron is dependent on the parameter of the infrared cutoff λ. We take it in the form $\lambda=m_e\alpha$ using the prescription $m_e\alpha^2\ll\lambda\ll m_e$ from Ref.[12].

Using the Fourier transform of the potential (73) and averaging the obtained expression over wave functions (4) we represent the electron vertex correction to the hyperfine splitting

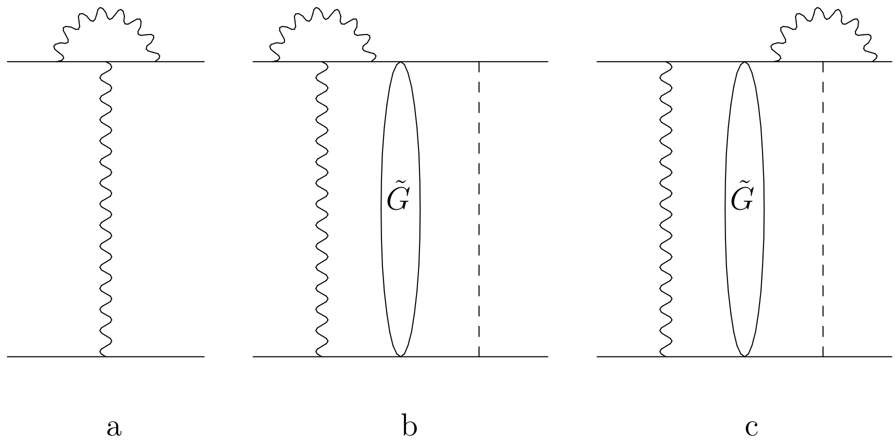


FIG. 8: The electron vertex corrections. The dashed line represents the Coulomb photon. The wave line represents the hyperfine part of the Breit potential. $\tilde{G}$ is the reduced Coulomb Green's function.

as follows:

$$b_{vertex,\ 1\gamma}=\nu_F\frac{\alpha}{32\pi^2}\left(\frac{M_e}{M_\mu}\right)\left(\frac{m_e}{\alpha M_\mu}\right)^3\int_0^\infty k^2dk\left[G_M^{(e)}(k^2)-1\right]\times \tag{74}$$

$$\times\left\{\left[1+\left(\frac{m_e}{4\alpha M_\mu}\right)^2k^2\right]\left[\left(\frac{M_e}{2M_\mu}\right)^2+\left(\frac{m_e}{4\alpha M_\mu}\right)^2k^2\right]^2\right\}^{-1}=4.218\ MHz.$$

Let us remark that the contribution (74) is of order α^5. Numerical value (74) is obtained after numerical integration with the one-loop expression of the electron magnetic form factor $G_M^{(e)}(k^2)$. If we use the value $G_M^{(e)}(k^2=0)$ then the electron vertex correction is equal 5.266 MHz. So, using the exact expression of the electron form factors in the one-loop approximation we observe the 1 MHz decrease of the vertex correction to the hyperfine splitting from 1γ interaction. Taking the expression (73) as an additional perturbation potential we have to calculate its contribution to HFS in the second order perturbation theory (see the diagram in Fig.8(b)). In this case the dashed line represents the Coulomb Hamiltonian ΔH (3). Following the method of the calculation formulated in previous section (see also Refs.[22, 23]) we divide again total contribution from the amplitude in Fig.8(b) into two parts which correspond to the muon ground state ($n=0$) and muon excited intermediate states ($n\neq 0$). In this way the first contribution with $n=0$ takes the form:

$$b_{vertex,\ SOPT}(n=0)=\frac{8\alpha^2}{3\pi^2m_em_\mu}\int_0^\infty k\left[G_M^{(e)}(k^2)-1\right]dk\int d\mathbf{x}_1\int d\mathbf{x}_3\psi_{e0}(\mathbf{x}_3)\times \tag{75}$$

$$\times\Delta\tilde{V}_1(k,\mathbf{x}_3)G_e(\mathbf{x}_1,\mathbf{x}_3)\Delta V_2(\mathbf{x}_1)\psi_{e0}(\mathbf{x}_1),$$

where $\Delta V_2(\mathbf{x}_1)$ is defined by Eq.(49) and

$$\Delta\tilde{V}_1(k,\mathbf{x}_3)=\int d\mathbf{x}_4\psi_{\mu0}(\mathbf{x}_4)\frac{\sin(k|\mathbf{x}_3-\mathbf{x}_4|)}{|\mathbf{x}_3-\mathbf{x}_4|}\psi_{\mu0}(\mathbf{x}_4)=\frac{\sin\left(\frac{kx_3}{4\alpha M_\mu}\right)}{x_3}\frac{1}{\left[1+\frac{k^2}{(4\alpha M_\mu)^2}\right]^2}. \tag{76}$$

Substituting the electron Green's function (27) in Eq.(75) we transform desired relation to the integral form:

$$b_{vertex,\ SOPT}(n=0) = \nu_F \frac{\alpha}{16\pi^2}\left(\frac{m_e}{\alpha M_\mu}\right)^2\left(\frac{M_e}{M_\mu}\right)^2\int_0^\infty \frac{k\left[G_M^{(e)}(k^2)-1\right]dk}{\left[1+\frac{m_e^2k^2}{(4\alpha M_\mu)^2}\right]^2}\times \tag{77}$$

$$\times\int_0^\infty x_3 e^{-\frac{M_e}{2M_\mu}x_3}\sin\left(\frac{m_e k}{4\alpha M_\mu}x_3\right)dx_3\int_0^\infty x_1\left(1+\frac{x_1}{2}\right)e^{-x_1\left(1+\frac{M_e}{2M_\mu}\right)}dx_1\times$$

$$\left[\frac{2M_\mu}{M_e x_>}-\ln(\frac{M_e}{2M_\mu}x_<)-\ln(\frac{M_e}{2M_\mu}x_>)+Ei(\frac{M_e}{2M_\mu}x_<)+\frac{7}{2}-2C-\frac{M_e}{4M_\mu}(x_1+x_3)+\frac{1-e^{\frac{M_e}{2M_\mu}x_<}}{\frac{M_e}{2M_\mu}x_<}\right]$$

$$=-0.208\ MHz.$$

One integration over the coordinate x_1 is carried out analytically and two other integrations are performed numerically. Second part of the vertex contribution (Fig.8(b)) with $n\neq 0$ can be reduced to the following form after several simplifications which are discussed in section II (see also Refs.[22, 23]):

$$b_{vertex,\ SOPT}(n\neq 0) = \nu_F\frac{8\alpha^4 M_e M_\mu^3}{\pi^3}\int e^{-2\alpha M_\mu x_2}d\mathbf{x}_2\int e^{-\alpha M_e x_3}d\mathbf{x}_3\int e^{-2\alpha M_\mu x_4}d\mathbf{x}_4\times \tag{78}$$

$$\times\int_0^\infty k\sin(k|\mathbf{x}_3-\mathbf{x}_4|)\left(G_M^{(e)}(k^2)-1\right)\frac{|\mathbf{x}_3-\mathbf{x}_2|}{|\mathbf{x}_3-\mathbf{x}_4|}\left[\delta(\mathbf{x}_4-\mathbf{x}_2)-\psi_{\mu 0}(\mathbf{x}_4)\psi_{\mu 0}(\mathbf{x}_2)\right].$$

We divide expression (78) into two parts as provided by two terms in the square brackets of (78). After that the integration (78) over the coordinates $\mathbf{x}_1$, $\mathbf{x}_3$ is carried out analytically. In the issue we obtain ($\gamma_2 = m_e k/4\alpha M_\mu$):

$$b_{1,vertex,\ SOPT}(n\neq 0) = \nu_F\frac{\alpha}{32\pi^2}\left(\frac{m_e}{\alpha M_\mu}\right)^3\frac{M_e}{M_\mu}\int_0^\infty k^2\left[G_M^{(e)}(k^2)-1\right]dk\frac{1}{(\gamma_1^2-1)^3}\times \tag{79}$$

$$\times\left[\frac{4\gamma_1(\gamma_1^2-1)}{(1+\gamma_2^2)^3}-\frac{\gamma_1(3+\gamma_1^2)}{(1+\gamma_2^2)^2}+\frac{4\gamma_1^2(\gamma_1^2-1)}{(\gamma_1^2+\gamma_2^2)^3}+\frac{1+3\gamma_1^2}{(\gamma_1^2+\gamma_2^2)^2}\right]=2.528\ MHz,$$

$$b_{2,vertex,\ SOPT}(n\neq 0) = -\nu_F\frac{\alpha}{32\pi^2}\left(\frac{m_e}{\alpha M_\mu}\right)^3\frac{M_e}{M_\mu}\int_0^\infty k^2\left[G_M^{(e)}(k^2)-1\right]dk\times \tag{80}$$

$$\times\frac{1}{(1+\gamma_2^2)^2}\left[\frac{2}{(\gamma_1^2+\gamma_2^2)}-\frac{(\gamma_1+1)}{[(1+\gamma_1)^2+\gamma_2^2]^2}-\frac{2}{(\gamma_1+1)^2+\gamma_2^2}-\frac{\gamma_2^2-3\gamma_1^2}{(\gamma_1^2+\gamma_2^2)^3}\right]=-0.831\ MHz.$$

It is necessary to emphasize that the theoretical error in the contributions $b_{1,2,vertex,SOPT}(n\neq 0)$ is determined by the factor $\sqrt{M_e/M_\mu}$ connected with the omitted terms of the expansion similar to Eq.(33) (see also Refs.[22, 23]). It can amount to 10% of the results (79), (80) that is the value near 0.2 MHz.

Until now we consider the electron vertex corrections connected with the hyperfine part of the interaction Hamiltonian (5). But in the second order perturbation theory we should

analyze vertex corrections to the Coulomb interactions of the electron and muon, electron and nucleus. Then in the coordinate representation we have the following potential:

$$\Delta V^C_{vertex,eN}(x_e)+\Delta V^C_{vertex,e\mu}(x_{e\mu})=\frac{2\alpha^2}{\pi^2}\int_0^\infty\frac{\left[G_E^{(e)}(k^2)-1\right]}{k}dk\left(\frac{\sin(kx_{e\mu})}{x_{e\mu}}-2\frac{\sin(kx_e)}{x_e}\right), \tag{81}$$

where we extract again the factor α/π from $\left[G_E^{(e)}(k^2)-1\right]$. $G_E^{(e)}$ is the electron electric form factor. One part of the contribution in Fig.8(c) is specified by the electron-muon intermediate states in which the muon is in the ground state $n=0$. This correction is determined by both terms in large parentheses of Eq.(81) and can be presented as follows:

$$b_{C,vertex,\ SOPT}(n=0)=\nu_F\frac{\alpha}{\pi^2}\left(\frac{M_e}{M_\mu}\right)^2\int_0^\infty x_3^2e^{-x_3\left(1+\frac{M_e}{2M_\mu}\right)}dx_3\times \tag{82}$$

$$\times\int_0^\infty x_1e^{-\frac{M_e}{2M_\mu}x_1}dx_1\int_0^\infty\frac{\left[G_E^{(e)}(k^2)-1\right]dk}{k}\sin\left(\frac{m_ek}{4\alpha M_\mu}x_1\right)\left\{1-\frac{1}{2\left[\frac{m_e^2k^2}{(4\alpha M_\mu)^2}+1\right]^2}\right\}\times$$

$$\left[\frac{2M_\mu}{M_ex_>}-\ln(\frac{M_e}{2M_\mu}x_<)-\ln(\frac{M_e}{2M_\mu}x_>)+Ei(\frac{M_e}{2M_\mu}x_<)+\frac{7}{2}-2C-\frac{M_e}{4M_\mu}(x_1+x_3)+\frac{1-e^{\frac{M_e}{2M_\mu}x_<}}{\frac{M_e}{2M_\mu}x_<}\right]$$

$$=-1.303\ MHz.$$

The index "C" means that the vertex correction to the Coulomb part of the Hamiltonian is considered. Excited states of the muon ($n\neq 0$) contribute to another part of the matrix element (Fig.8(c)). Changing the Coulomb Green's function of the electron by free Green's function (see discussion in section II) we can carry out the coordinate integration and express the correction to HFS as one-dimensional integral:

$$b_{C,vertex,\ SOPT}(n\neq 0)=-\nu_F\frac{8\alpha}{\pi^2}\frac{M_e}{M_\mu}\left(\frac{\alpha M_\mu}{m_e}\right)\int_0^\infty\frac{\left[G_E^{(e)}(k^2)-1\right]dk}{k^2}\left\{1-\frac{1}{\left[1+\frac{m_e^2k^2}{(4\alpha M_\mu)^2}\right]^4}\right\}= \tag{83}$$

$$=1.806\ MHz.$$

The electron vertex corrections investigated in this section have the order α^5 in the hyperfine interval. Summary value of all obtained contributions (74), (77), (79), (80), (82), (83) is equal to 6.210 MHz. It differs by a significant value 0.944 MHz from the result 5.266 MHz which was used previously by many authors for the estimation of the electron anomalous magnetic moment contribution. On our opinion, it is necessary to use the same approach for the calculation of the electron vertex corrections by the variational method [19] in which the bound state wave function has the form $\psi(\mathbf{x}_e,\mathbf{x}_\mu,\mathbf{x}_{e\mu})=\sum_i C_ie^{-\alpha_ix_e-\beta_ix_\mu-\gamma_ix_{e\mu}}$ and the parameters α_i, β_i, γ_i are chosen randomly between some minimal and maximal values.

V. CONCLUSIONS

In the present study, we have performed the analytical and numerical calculation of several important contributions to the hyperfine splitting of the ground state in muonic helium atom

TABLE I: Hyperfine splitting of the ground state in the muonic helium atom ($\mu\ e\ ^3_2He$).

Contribution to the HFS	b, MHz	c, MHz	Reference
The Fermi splitting	4516.307	1083.256	(11), (13), [1, 22, 23]
Recoil correction of order $\alpha^4(m_e/m_\mu)$	-64.322	8.323	(12), (14), [1]
Correction of muon anomalous magnetic moment of order α^5	5.266	—	(11), [1, 22]
Recoil correction of order $\alpha^4(M_e/m_\alpha)\sqrt{(M_e/M_\mu)}$	0.105	—	[22, 23]
Relativistic correction of order α^6	0.040	0.087	[2, 5]
One-loop VP contribution in 1γ interaction of orders α^5,α^6	0.036	0.021	(23), (24)
One-loop VP contribution in the second order PT	0.062	-0.023	(26),(28),(31),(35),(36),(38), (39),(44),(45),(51),(55),(57)
Nuclear structure correction in 1γ interaction of order α^6	—	-0.072	(60)
Nuclear structure and recoil correction in 2γ interactions of order α^5	—	-0.077	(63)
Nuclear structure correction of order α^6 in second order PT	-0.013	-0.074	(14),(65),(66)
Recoil correction of order $\alpha^5(m_e/m_\mu)\ln(m_e/m_\mu)$	0.812	—	(72) ,[5, 37]
Electron vertex correction of order α^6	-0.615	-0.035	[13, 38–40]
Electron vertex contribution of order α^5	6.210	—	(74),(77),(79), (80),(82),(83)
Summary contribution	4463.888	1091.406	$\Delta\nu^{hfs} = 4166.471$ MHz

connected with the vacuum polarization, the nuclear structure, recoil effects and the electron vertex corrections. To solve this task we use the method of the perturbation theory which was formulated previously for the description of the muonic helium hyperfine splitting in Refs.[1, 22, 23]. We have considered corrections of the electron vacuum polarization, electron electromagnetic form factors and the nuclear structure effects of orders α^5 and α^6. The numerical values of the corresponding contributions are displayed in Table I. We present in Table I the references to the calculations of other corrections which are not considered here. The relativistic correction was obtained in Ref.[3, 5], the electron vertex corrections of order $\alpha(Z\alpha)E_F$ were calculated in the case of hydrogenic atoms in Refs.[13, 38–40]. Basic contributions to the hyperfine splitting obtained by Lakdawala and Mohr are also included in Table I because our calculation is closely related to their approach.

Let us list basic points related to the calculation.

1. For muonic helium atom, the vacuum polarization effects are important for obtaining the high accuracy of the calculation. They give rise to the modification of the two-particle

interaction potential which provides the $\alpha^5\frac{M_e}{M_\mu}$-order corrections to the hyperfine structure. The next to leading order vacuum polarization corrections (two-loop vacuum polarization) are negligible.

2. The electron vertex corrections to the coefficient b should be considered with the exact account of the one-loop electromagnetic form factors of the electron because the characteristic momentum incoming in the electron vertex operator is of order of the electron mass.

3. The nuclear structure corrections to the ground state hyperfine splitting are expressed in terms of electromagnetic form factors and the charge radius of the helion.

4. Analyzing the one-loop electron vacuum polarization and vertex effects and the nuclear structure contributions in each order in α, we have taken into account recoil terms proportional to the ratio of the electron and muon masses.

The resulting numerical value 4466.471 MHz of the smaller ground state hyperfine splitting in muonic helium ($\mu\ e\ ^3_2He$) is presented in Table I. It is sufficiently close both to the experimental result (1) and the earlier performed calculations by the perturbation theory, variational approach in [1, 2, 5]. The estimation of the theoretical uncertainty can be done in terms of the Fermi energy ν_F and small parameters α and the ratio of the particle masses. On our opinion, there exist several main sources of the theoretical errors. First of all, as we mentioned above comprehensive analytical and numerical calculation of recoil corrections of orders $\alpha^4\frac{M_e}{M_\mu}$, $\alpha^4\frac{M_e^2}{M_\mu^2}$, $\alpha^4\frac{M_e^2}{M_\mu^2}\ln(M_\mu/M_e)$ was carried out by Lakdawala and Mohr in the second order PT in Refs.[22, 23]. The error of their calculation connected with the correction $\nu_F\frac{M_e^2}{M_\mu^2}\ln\frac{M_\mu}{M_e}$ consists 0.6 MHz. The second source of the error is related to contributions of order $\alpha^2\nu_F \approx 0.2$ MHz which appear both from QED amplitudes and in higher orders of the perturbation theory. Another part of the theoretical error is determined by the two-photon three-body exchange amplitudes mentioned above. They are of the fifth order over α and contain the recoil parameter $(m_e/m_\alpha)\ln(m_e/m_\alpha)$, so that their possible numerical value can be equal ± 0.05 MHz. Finally, a part of theoretical error is connected with our calculation of the electron vertex corrections of order α^5 in section IV. It consists at least 0.2 MHz (see the discussion after Eq.(80)). We neglect also the electron vertex contributions of order $\nu_F\alpha M_e/M_\mu \approx 0.2$ MHz which appear in higher orders of the perturbation theory. Thereby, the total theoretical uncertainty is not exceeded ± 0.7 MHz. The existing difference between the obtained theoretical result and experimental value of the hyperfine splitting (1) equal to 0.171 MHz lies in the range of total error. Theoretical error which remains sufficiently large in the comparison with the experimental uncertainty, initiates further theoretical investigation of the higher order recoil contributions and more careful construction of the three-particle interaction operator connected with the multiphoton exchanges.

Acknowledgments

This work was supported by the Federal Program "Scientific and pedagogical personnel of innovative Russia under Grant No. NK-20P/1.

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[1] S. D. Lakdawala and P. Mohr, Phys. Rev. A **24**, 2224 (1981).

[2] K.-N. Huang and V. W. Hughes, Phys. Rev. A **26**, 2330 (1982).
[3] K.-N. Huang and V. W. Huges, Phys. Rev. A **20**, 706 (1979).
[4] R. L. Drachman, J. Phys. B **16**, L749 (1983).
[5] M.-K. Chen, J. Phys. B **26**, 2263 (1993).
[6] M.-K. Chen, J. Phys. B **23**, 4041 (1990).
[7] V. L. Yakhontov and M. Ya. Amusia, J. Phys. B **27**, 3743 (1994)
[8] R. Krivec and V. B. Mandelzweig, Phys. Rev. **57**, 4976 (1998).
[9] A. M. Frolov, Phys. Rev. A **61**, 022509 (2000).
[10] R. Pohl, A. Antognini, F. Nez et al., Nature **466**, 213 (2010).
[11] M. Gladish et al., Proc. 8th Int. Conf. on Atom. Phys., ed. I. Lindgren, A. Rosen and S. Svanberg, Plenum, New York, 1983.
[12] H. A. Bethe and E. E. Salpeter, Quantum mechanics of one- and two-electron atoms, Springer, Berlin, 1957.
[13] M. I. Eides, H. Grotch and V. A. Shelyuto, Phys. Rep. **342**, 62 (2001).
[14] S. G. Karshenboim, Phys. Rep. **422**, 1 (2005).
[15] R. N. Faustov and A. P. Martynenko, Yad. Fiz. **45**, 770 (1987) [Sov. J. Nucl. Phys. **45**, 479 (1987)]; Phys. Atom. Nucl. **66**, 1719, (2003); JETP **88**, 672, (1999); JETP **98**, 39 (2004)
[16] A. P. Martynenko, Phys. Rev. A **76**, 012505 (2007).
[17] A. P. Martynenko, JETP **106**, 691 (2008); arXiv:0710.3237 [hep-ph].
[18] A. M. Frolov, Phys. Rev. A **57**, 2436 (1998).
[19] V. I. Korobov, Phys. Rev. A **61**, 064503 (2000).
[20] K. Pachucki, Phys. Rev. A **63**, 032508 (2001).
[21] A. A. Krutov and A. P. Martynenko, Phys. Rev. A **78**, 032513 (2008).
[22] S. D. Lakdawala and P. J. Mohr, Phys. Rev. A **22**, 1572 (1980).
[23] S. D. Lakdawala and P. J. Mohr, Phys. Rev. A **29**, 1047 (1984).
[24] E. Borie, Z. Phys. A **291**, 107 (1979).
[25] P. J. Mohr, B. N. Taylor and D. B. Newell, Rev. Mod. Phys. **80**, 633 (2008).
[26] V. B. Berestetskii, E. M. Lifshits and L. P. Pitaevskii, Quantum Electrodynamics, Nauka, Moscow, 1980.
[27] A. P. Martynenko, Phys. Rev. A **71**, 022506 (2005).
[28] A. P. Martynenko, JETP **101**, 1021 (2005); hep-ph/0412250.
[29] E. N. Elekina, A. P. Martynenko, arXiv:0909.2759 [hep-ph].
[30] H. F. Hameka, Jour. Chem. Phys. **47**, 2728 (1967).
[31] V. A. Fok, Principles of Quantum Mechanics, Nauka, Moscow, 1976.
[32] H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969).
[33] G. T. Bodwin and D. R. Yennie, Phys. Rev. D **37**, 498 (1988).
[34] R. N. Faustov and A. P. Martynenko, Phys. Atom. Nucl. **65**, 265 (2002); Phys. Atom. Nucl. **63**, 845 (2000); Phys. Rev. A **67**, 052506 (2003).
[35] J. S. McCarthy, I. Sick and R. R. Whitney, Phys. Rev. C **15**, 1396 (1977).
[36] D. C. Morton, Q. Wu and G. W. F. Drake, Phys. Rev. C **73**, 034502 (2006).
[37] R. Arnowitt, Phys. Rev. **92**, 1002 (1953).
[38] S. J. Brodsky and G. W. Erickson, Phys. Rev. **148**, 26 (1966).
[39] N. Kroll and F. Pollack, Phys. Rev. **84**, 594 (1951).
[40] R. Karplus, A. Klein and J. Schwinger, Phys. Rev. **84**, 597 (1951).